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1987 J. Phys. A: Math. Gen. 20 4027

(http://iopscience.iop.org/0305-4470/20/12/046)

J. Phys. A: Math. Gen. 20 (1987) 4027-4030. Printed in the UK

COMMENT

Singular point analysis, resonances and Yoshida's theorem

W-H Steeb†, J A Louw† and M F Maritz‡

† Department of Physics, Rand Afrikaans University, PO Box 524, Johannesburg 2000, South Africa
‡ Department of Applied Mathematics, University of the Orange Free State, Bloemfontein 9300, South Africa

Received 9 January 1987

Abstract. Yoshida described a connection between scale-invariant autonomous systems of ordinary differential equations, algebraic first integrals and resonances. In his analysis it is assumed that the scale invariance determines the dominant behaviour. Here we discuss the case where the dominant behaviour is not determined by the scale invariance. For the constructed example we also give the Lax representation.

Many authors (cf Steeb *et al* 1985, Steeb and Louw 1986 and references therein) have investigated dynamical systems with the help of the singular point analysis (also called the Painlevé test) in order to find out whether or not the systems are integrable. In most cases the Painlevé test has been applied as a recipe. If a given differential equation passes the Painlevé test, then in many cases the equation is integrable and one can try to find the first integrals in order to integrate the equation.

Recently, Yoshida (1983a,b) derived a connection between scale-invariant autonomous systems of ordinary differential equations of first order

$$\dot{u}_j = F_j(u) \tag{1}$$

($j = 1, \ldots, n$) algebraic first integrals and resonances (Kowalevski's exponents). Here $F_j(u)$ is a rational function of $u_1, \ldots, u_n$. The starting point in Yoshida's analysis is the scale invariance of system (1) under

$$t \to \varepsilon^{-1} t \qquad u_1 \to \varepsilon^{g_1} u_1 \qquad \ldots \qquad u_n \to \varepsilon^{g_n} u_n \tag{2}$$

for a set of rational numbers $g_1, \ldots, g_n$. Then he assumed that system (1) admits a non-trivial solution of the form $u_j(t) = a_{j0}\tau^{-g_j}$ where $\tau \equiv t - t_1$. With these assumptions he gave the following theorem.

Theorem. Let $I(u)$ be a weighted homogeneous algebraic first integral of weighted degree r for the similarity invariant system (1). Suppose that the elements of the vector grad $I(a_0)$ are finite and not identically zero for a fixed choice of the set $a_{10}, \ldots, a_{n0}$ in $F_j(a_{10}, \ldots, a_{n0}) = -g_j a_{j0}$. Then r is a resonance.

Now there are many systems of interest which are scale invariant under the transformation (2). Nevertheless they do not admit non-trivial solutions $u_j(t) = a_{j0}\tau^{-g_j}$ ($j = 1, \ldots, n$), i.e. one finds $a_{j0} = 0$ for $j = 1, \ldots, n$. This case has not been studied by Yoshida (1983a,b). In the present comment we discuss this case. To this purpose we construct a hierarchy of scale-invariant autonomous systems of ordinary differential equations. This class belongs to the Lotka–Volterra models. The Painlevé test will be performed. Furthermore we give the Lax representation $L_t = [A, L]$. The first integrals are given by $\mathrm{Tr}(L^k)$ $(k = 1, 2, \ldots)$. It turns out that the system is algebraically completely integrable.

We start from the partial differential equation $U_t + \sigma U U_x = 0$ with initial condition $U(x, 0) = U_0(x)$ and periodic boundary condition $U(0, t) = U(1, t)$. The quantity σ is a positive constant. This equation is well known in literature. To obtain an autonomous system of first-order differential equations we perform the semidiscretisation $U(hj, t) \to u_j(t)$. This yields

$$\dot{u}_j + u_j\left(\frac{1-\theta}{2h}(u_{j+1} - u_j) + \frac{1+\theta}{2h}(u_j - u_{j-1})\right) = 0 \tag{3}$$

with $j = 1, \ldots, n$ $(n \geqslant 3)$, $-1 \leqslant \theta \leqslant 1$ and cyclic boundary conditions $0 \equiv n$, $n+1 \equiv 1$. In the following we put $\theta = 0$ and perform the scaling such that $t \to -\sigma t/2h$. Then we arrive at

$$\begin{aligned} \dot{u}_1 &= u_1(u_2 - u_n) \\ \dot{u}_2 &= u_2(u_3 - u_1) \\ &\vdots \\ \dot{u}_n &= u_n(u_1 - u_{n-1}). \end{aligned} \tag{4}$$

For this system we discuss the singular point analysis and Yoshida's theorem given above.

First of all we notice that system (4) is scale invariant under $t \to \varepsilon^{-1} t$, $u_j \to \varepsilon u_j (j = 1, \ldots, n)$. Owing to the cyclic boundary conditions we find that

$$I_1(u) = \sum_{j=1}^{n} u_j \tag{5}$$

and

$$I_2(u) = \prod_{j=1}^{n} u_j \tag{6}$$

are polynomial first integrals of system (4). For $n \geqslant 4$ we obtain the further first integral

$$I_3(u) = \sum_{j=1}^{n} u_{j-1} u_{j+1}. \tag{7}$$

For a given n we find $n-1$ polynomial first integrals and thus the system (4) is algebraically completely integrable.

Let us first discuss the case $n = 3$ in detail. The Lax representation is given by

$$L = \begin{pmatrix} 0 & 1 & u_1 \\ u_2 & 0 & 1 \\ 1 & u_3 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} u_1 + u_2 & 0 & 1 \\ 1 & u_2 + u_3 & 0 \\ 0 & 1 & u_1 + u_3 \end{pmatrix}. \tag{8}$$

The first integrals are given by $\mathrm{Tr}\, L^2$ and $\mathrm{Tr}\, L^3$. Let us now perform the Painlevé test. Owing to the scale invariance we are motivated to try the ansatz $u_j(t) = a_{j0}\tau^{-1}$ where $j = 1, 2, 3$ and $a_{j0} \neq 0$. Inserting this ansatz into system (4) leads to

$$-a_{10} = a_{10}(a_{20} - a_{30}) \qquad -a_{20} = a_{20}(a_{30} - a_{10}) \qquad -a_{30} = a_{30}(a_{30} - a_{20}).$$

Consequently, we only find the trivial solution $a_{10}=a_{20}=a_{30}=0$. Thus the ansatz motivated from the scale invariance does not work. A successful ansatz for the dominant behaviour is given by

$$u_1(t)\propto a_{10}\tau^{-1} \qquad u_2(t)\propto a_{20}\tau^{-1} \qquad u_3(t)\propto a_{30}\tau^2. \tag{9}$$

Since system (4) with $n=3$ is invariant under $u_1\to u_2$, $u_2\to u_3$, $u_3\to u_1$ we find more than one branch. However, without loss of generality we can restrict our consideration to branch (9). Inserting ansatz (9) into system (4) we obtain the system with the dominant terms

$$\dot{u}_1=u_1u_2 \qquad \dot{u}_2=-u_1u_2 \qquad \dot{u}_3=u_3(u_1-u_2) \tag{10}$$

where $a_{10}=1$, $a_{20}=-1$ and a_{30} is arbitrary. The first integrals of system (10) are given by $I_1(u)=u_1+u_2$ and $I_2(u)=u_1u_2u_3$. System (10) is scale invariant under $t\to\varepsilon^{-1}t$, $u_1\to\varepsilon u_1$, $u_2\to\varepsilon u_2$, $u_3\to\varepsilon^{\alpha}u_3$ where α is arbitrary ($\alpha\neq 0$). The resonances of system (10) are given by $r_1=-1$, $r_2=0$ (related to a_{30}) and $r_3=1$. Due to the theorem of Yoshida the first integral I_1 corresponds to the resonance $r_3=1$ since $I_1(\varepsilon u_1,\varepsilon u_2,\varepsilon^{\alpha}u_3)=\varepsilon I_1(u_1,u_2,u_3)$. On the other hand, we find $I_2(\varepsilon u_1,\varepsilon u_2,\varepsilon^{\alpha}u_3)=\varepsilon^{2+\alpha}I_2(u_1,u_2,u_3)$. Thus this first integral can only be associated with the resonance $r_2=0$ when $\alpha=-2$.

Inserting the Laurent expansion

$$u_j(t)=\sum_{k=0}^{\infty}a_{jk}\tau^{-1+k}$$
$$u_3(t)=\sum_{k=0}^{\infty}a_{3k}\tau^{2+k} \tag{11}$$

where $j=1,2$, we find that system (4) ($n=3$) passes the Painlevé test.

Before we study the general case let us discuss the case $n=4$. The Lax pair L and A is given by

$$L=\begin{pmatrix}0&1&0&u_1\\u_2&0&1&0\\0&u_3&0&1\\1&0&u_4&0\end{pmatrix} \qquad A=\begin{pmatrix}u_1+u_2&0&1&0\\0&u_2+u_3&0&1\\1&0&u_3+u_4&0\\0&1&0&u_4+u_1\end{pmatrix}. \tag{12}$$

It is interesting to note that $\operatorname{Tr}L=0$, $\operatorname{Tr}L^2=2I_1$, $\operatorname{Tr}L^3=0$, $\operatorname{Tr}L^4=4+4I_3+2I_1^2-4I_2$. Inserting the ansatz $u_j(t)=a_{j0}\tau^{-1}$ leads to the trivial solution $a_{10}=\ldots=a_{40}=0$. A successful ansatz for the dominant behaviour is given by

$$u_j(t)\propto a_{j0}\tau^{-1}$$
$$u_k(t)\propto a_{k0}\tau \tag{13}$$

where $j=1,2$ and $k=3,4$. Notice that there are other branches since system (4) ($n=4$) is invariant under $u_1\to u_2$, $u_2\to u_3$, $u_3\to u_4$, $u_4\to u_1$. However, without loss of generality we can restrict our consideration to ansatz (13). Inserting ansatz (13) into (4) ($n=4$) we find the system with the dominant terms, namely

$$\dot{u}_1=u_1u_2 \qquad \dot{u}_2=-u_2u_1 \qquad \dot{u}_3=-u_3u_2 \qquad \dot{u}_4=u_4u_1 \tag{14}$$

where $a_{10}=1$, $a_{20}=-1$ and a_{30}, a_{40} are arbitrary. The first integrals of system (14) are given by $I_1(u)=u_1+u_2$, $I_2(u)=u_1u_2u_3u_4$ and $I_3(u)=u_1u_3+u_2u_4$. For the resonances we obtain $r_1=-1$, $r_2=0$ (twofold) and $r_3=1$. The resonance $r_2=0$ (twofold) is related to the arbitrariness of a_{30} and a_{40}. Again I_1 is related to the resonance $r_3=1$. System

(14) is scale invariant under $u_1 \to \varepsilon u_1$, $u_2 \to \varepsilon u_2$, $u_3 \to \varepsilon^\alpha u_3$ and $u_4 \to \varepsilon^\beta u_4$ where α and β are arbitrary ($\alpha, \beta \neq 0$). Then $I_2(\varepsilon u_1, \varepsilon u_2, \varepsilon^\alpha u_3, \varepsilon^\beta u_4) = \varepsilon^{2+\alpha+\beta} I_2(u_1, u_2, u_3, u_4)$ and $I_3(\varepsilon u_1, \varepsilon u_2, \varepsilon^\alpha u_3, \varepsilon^\beta u_4) = \varepsilon^{1+\alpha} u_1 u_3 + \varepsilon^{1+\beta} u_2 u_4$. We find that system (4) with $n = 4$ passes the Painlevé test.

For arbitrary n the Lax representation is given by

$$\begin{aligned} &\phi_{j+1} + u_j \phi_{j-1} = \lambda \phi_j \\ &\dot{\phi}_j = \phi_{j+2} + (u_{j+1} + u_j)\phi_j \end{aligned} \tag{15}$$

where $j = 1, \ldots, n$ and $0 \equiv n$, $1 \equiv n+1$, etc. The parameter λ is time independent. As described already for $n = 3$ and $n = 4$ the ansatz $u_j(t) = a_{j0}\tau^{-1}$ gives only the trivial solution. Consequently, not every term in system (4) can be dominant. The ansatz $u_j(t) \propto a_{j0}\tau^{k_j}$ to find the dominant behaviour leads to positive and negative integers for $(k_1, \ldots, k_n)$. Owing to the discrete symmetry $u_1 \to u_2, \ldots, u_n \to u_1$ of system (4) we find more than one branch. We must distinguish between n odd and n even. Then it can be shown that system (4) passes the Painlevé test for arbitrary n. Not all first integrals correspond to resonances.

To summarise: the constructed hierarchy of equations shows that the theorem of Yoshida which connects algebraic first integrals and resonances cannot be applied, in general, to scale-invariant autonomous systems of ordinary differential equations. The method fails in the constructed example since, from the scale-invariant system (4), it does not follow that the dominant behaviour is given by $u_j(t) = a_{j0}\tau^{-1}$ with $a_{j0} \neq 0$.

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